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Renormalization Group Approach to Random Walks on Disordered Lattices

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One-dimensional random walks with static disorder are analyzed using a real space renormalization group procedure. The presence of disorder leads to a non-Markovian description of the macroscopic behavior of the random walk. We consider random walks with nearest-neighbor hopping described by a master equation with both on-site and site-to-site disorder in the transition matrix. Site-to-site disorder leads to a generalized diffusion coefficient with a $t^{-3/2}$ long time tail whereas on-site disorder leads to a generalized Burnett coefficient with a $t^{-1/2}$ long time tail.

KEY WORDS: Random walks; real space renormalization group; long time tails; one-dimension; disorder.

1. INTRODUCTION

In this article I will discuss the macroscopic properties of one-dimensional random walks with static disorder. The main points of the article are: Firstly, that the presence of disorder leads to interesting non-Markovian features in the macroscopic description of the random walk and, secondly, that a real space renormalization group procedure can be used to analyze the effect of strong disorder.

The model that I will treat can be described by a master equation of the form

$$\frac{d}{dt} P_n(t) = \sum_m T_{nm} P_m(t) \tag{1}$$

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$P_n(t)$ is the probability that the walker is at the nth site at time t. The transition matrix, T_{nm}, allows only nearest-neighbor hopping

$$T_{nm} = \frac{W_m}{C_m}\delta_{nm+1} + \frac{W_{m-1}}{C_m}\delta_{nm-1} - \left(\frac{W_m + W_{m-1}}{C_m}\right)\delta_{nm} \tag{2}$$

The sites are evenly spaced on a line with spacing l.

The parameters W_n and C_n which determine the transition matrix are positive numbers which vary randomly from site to site but do not change with time. The values of C_n and W_n are independently chosen at each site and identically distributed from site to site. The distributions for C and W^{-1} must have finite first and second moments. For a physical picture of the meaning of these parameters imagine that the random walk represents the motion of a particle in a disordered one-dimensional potential with equally spaced valleys. C_n is then related to the depth of the nth valley while W_n is related to the height of the barrier separating the nth and $(n+1)$th valley.

The model discussed here is a generalization of a model in which the W's are random but C is a constant. The long time properties of this model have been analyzed by a variety of methods[1–5] including an exact calculation by Zwanzig.[4] All of these calculations are in agreement with one another. The interesting feature found in this random walk is that its macroscopic behavior must be described by a non-Markovian diffusion equation with a memory kernel that decays for long times like $t^{-3/2}$. If the random walk models a one-dimensional conductor, then randomness in the W's leads to a square root of frequency behavior in the low-frequency conductivity at low frequencies.

In a previous paper,[1] I showed how to obtain this result using a renormalization group method. In the present article this method is extended to the more general case where both the W's and the C's are random. The new feature which we will find in this case is that the generalized diffusion equation describing the macroscopic relaxation of the system now takes a more complicated form which includes a non-Markovian Burnett (∇^4) term with a memory kernel which decays like $t^{-1/2}$. On the other hand, randomness in the C's has no effect on the diffusion (∇^2) part of the generalized diffusion equation and, therefore, has no effect on the frequency-dependent conductivity of the system.

In the next section the macroscopic response function is defined and calculated for a system with weak disorder using perturbation theory. In the third section the result is extended to the case of strong disorder using the renormalization group method introduced in Ref. 1. The paper closes with a brief discussion.

2. THE RESPONSE FUNCTION FOR A WEAKLY DISORDERED SYSTEM

The viewpoint that I take in analyzing the system is that of response theory. Imagine that each member of a collection of systems is allowed to equilibrate in the same external potential, U_m, which is then suddenly turned off at $t = 0$. How does this ensemble relax from its initial, local equilibrium state to its final, total equilibrium state?

To answer that question it is convenient to use Laplace transforms. Let $\tilde{P}_m(z)$ be defined by

$$\tilde{P}_m(z) = \int_0^\infty e^{-zt} P_m(t)\, dt \tag{3}$$

The formal solution of the master equation is

$$\tilde{P}_m(z) = \sum_n G_{mn}(z) P_n(0) \tag{4}$$

where the Green's function is given, in matrix form, by

$$G(z) = \frac{1}{z - T} \tag{5}$$

The initial condition we envision is local equilibrium in the presence of the field, U. Thus

$$P_m(0) = C_m e^{-U_m/k\mathscr{T}} \tag{6}$$

where $\mathscr{T}$ is the temperature and k is Boltzmann's constant. Averaging over the distributions for W and C (indicated by $\langle\ \rangle$) with this initial condition yields the following equation for the mean relaxation of the perturbed ensemble:

$$\langle \tilde{P}_m(z) \rangle = \sum_n \mathscr{F}_{mn}(z) \langle P_n(0) \rangle \tag{7}$$

where the response function, $\mathscr{F}$ is given by

$$\mathscr{F}_{mn}(z) = \langle G_{mn}(z) C_n \rangle / c \tag{8}$$

with

$$c = \langle C \rangle \tag{9}$$

Notice that in the case where the C's are not random the response function can be identified with the average Green's function. In the case when the C's are random the response function behaves in a quite different way than the average Green's function. The latter quantity corresponds to the relaxation of an ensemble in which the initial occupation probabilities

$P_m(0)$, have been fixed for each member of the ensemble independently of the equilibrium probabilities, C_m. The response function describes the relaxation of a system from a local equilibrium state where the initial deviation from equilibrium is the same for each member of the ensemble. In accordance with a macroscopic viewpoint, I will focus primarily on the response function and make only brief mention of the behavior of the average Green's function in the discussion.

To make use of the translational invariance of the ensemble it is convenient to use Fourier transforms. Let

$$P(q,z) = \sum_m \tilde{P}_m(z) e^{iqml} \tag{10}$$

which for each member of the ensemble satisfies

$$P(q,z) = \int dq'\, G(q,q';z) P(q', t=0) \tag{11}$$

where the limits of integration are taken from $-\pi/l$ to π/l and the Fourier transform of the Green's function is

$$G(q,q';z) = \frac{l}{2\pi} \sum_{nm} e^{iqln - iq'lm} G_{nm}(z) \tag{12}$$

Similarly, the mean relaxation of the ensemble from a local equilibrium state is given by the Fourier transformed response function, $\mathscr{F}_q(z)$

$$\langle P(q,z)\rangle = \mathscr{F}_q(z) \langle P(q,t=0)\rangle \tag{13}$$

with

$$\mathscr{F}_q(z)\delta(q-q') = \int dq''\, \langle G(q,q'+q'';z) C(q'')\rangle / c \tag{14}$$

For the case of a uniform system defined by the parameters w, c, and l the Green's function can be obtained immediately from Eqs. (2), (5), and (12):

$$G(q,q';z) = G_q(z)\delta(q-q') = \frac{c\delta(q-q')}{zc + 4w\sin^2(ql/2)} \tag{15}$$

Now consider a collection of weakly disordered systems. We will use perturbation theory and expand the response function around a uniform system defined by $w = \langle W\rangle$ and $c = \langle C\rangle$ in powers of the deviations from the average values $\delta W = W - w$ and $\delta C = C - c$. The desired result can be obtained from a functional Taylor series expansion for the Green's function in powers of ΔT_{nm}, the deviation of T_{nm} from its value in a uniform system

with $W = w$ and $C = c$,

$$\mathscr{F}_q \delta(q - q') = G_q \delta(q - q') + G_q^2 \langle \Delta T(q, q') \rangle + \frac{l}{2\pi c} \int dq'' \, G_q G_{q+q''} \langle \Delta T(q, q' + q'') \delta C(q'') \rangle + \int dq'' \, G_q^2 G_{q''} \langle \Delta T(q, q'') \Delta T(q'', q') \rangle + \cdots \tag{16}$$

Equation (16) follows from Eq. (5), (12), and (14).

Rather than evaluating the response function itself we shall evaluate a generalized transport coefficient defined by

$$\mathscr{F}_q(z) = \frac{1}{z + q^2 U(q, z)} \tag{17}$$

After a straightforward but lengthy calculation, the following expression for $U(q, t)$ is obtained

$$U(q, z) = D \left\{ 1 - \frac{\langle \delta W^2 \rangle}{w^2} + \left(\frac{zc}{4w} \right)^{1/2} \frac{\langle \delta W^2 \rangle}{w^2} \right\} - D \left(\frac{q^2 D}{z} \right) \left(\frac{zc}{4w} \right)^{1/2} \frac{\langle \delta C^2 \rangle}{c^2} \tag{18}$$

where the diffusion coefficient, D, is given by

$$D = wl^2/c \tag{19}$$

This expression is valid to second order in the fluctuating quantities δW and δC and to order $z^{1/2}$ for small z holding q^2/z fixed.

The first term on the right-hand side of Eq. (18) is a generalized diffusion coefficient with a $z^{1/2}$ or $t^{-3/2}$ long time tail. The second term is a generalized Burnett coefficient with a $z^{-1/2}$ or $t^{-1/2}$ long time tail. This result is mathematically and physically closely related to the long time tails found in fluids (7) and the Lorentz gas (8).

3. RENORMALIZATION GROUP APPROACH

The results of second-order perturbation theory are valid only for the case of weak disorder when higher powers of δC and δW can be ignored. In the case of strong disorder we can use the renormalization group (RG) to transform the original problem into a new problem for which perturbation theory can be used exactly in the limit of small q and z. The method is

similar to that used by Forster *et al.*[9] in their study of randomly stirred fluids.

The first step in the renormalization group transformation is to eliminate odd-numbered lattice sites according to the following rule: If the walker is at an even numbered site in the original process then it is at the same site in the new process. If the walker is at an odd-numbered site in the original process then, in the new process, it is defined to be at the even-numbered site which it most recently visited. The next step in the RG transformation is to rescale lengths and times and renumber lattice sites so that the new process looks as much as possible like the old process. Lengths and lattice numbers must be divided by 2 so that the new lattice spacing is the same as the old and so that the lattice is again numbered by all the integers. For the moment, let the time be rescaled by an arbitrary factor, $1/\lambda$. In the Appendix, recursion relations are obtained which express the parameters describing the new process in terms of the original set of C's and W's. Denoting the new set of parameters with a prime, the result is

$$\frac{1}{W_n'} = \frac{1}{2}\left(\frac{1}{W_{2n}} + \frac{1}{W_{2n+1}}\right) \tag{20}$$

$$C_n' = \frac{2}{\lambda}\left[C_{2n} + \frac{1}{2}(C_{2n-1} + C_{2n+1}) + \frac{1}{2}\left(\frac{W_{2n-2} - W_{2n-1}}{W_{2n-2} + W_{2n-1}}\right)C_{2n-1} - \frac{1}{2}\left(\frac{W_{2n} - W_{2n+1}}{W_{2n} + W_{2n+1}}\right)C_{2n+1}\right] \tag{21}$$

We choose $\lambda = 4$, so that $c = \langle C \rangle$ will be unaffected by the RG transformation. The choice of $\lambda = 4$ is the natural one for time rescaling in a diffusive process where the mean squared displacement grows linearly with the time.

The strategy is to iterate the RG transformation may times and then to use perturbation theory to calculate $U(q,z)$ from the parameters $C_m^{(N)}$ and $W_m^{(N)}$ of the Nth RG iterate of the original process. Before carrying this out, let us consider the errors introduced by the RG transformation. By eliminating odd-numbered sites an uncertainty of at most one lattice spacing is introduced in the position of the walker. An additional error arises from the fact that the set $\{C_m^{(N)}, W_m^{(N)}\}$ no longer gives a complete description of the transformed process. The reason is that, after the elimination of the odd-numbered sites, the waiting time for jumping between even-numbered sites is no longer exponential so that the process obeys a non-Markovian master equation described by a set of functions $\{C_m^{(N)}(z), W_m^{(N)}(z)\}$. The recursion relations for these functions are given in the Appendix. The z dependence of these functions is analytic, so, to order $z^{1/2}$ we can work

with the truncated description embodied in the set $C_m^{(N)}, W_m^{(N)}$. Thus, for small q and z we can accurately calculate $U(q,z)$ using the parameters of the Nth iterate of the RG so long as we take the space and time rescaling into account,

$$U(q,z) = U^{(N)}(2^N q, 4^N z) \tag{22}$$

We now apply perturbation theory to the right-hand side of Eq. (22). One technical difficulty arises here. The recursion relation for the C's introduces correlations between nearest-neighbor C's and between the C's and W's. The correlations between the C's and W's do not appear in the perturbation theory to the order we are working but the correlation of neighboring C's introduces an extra term on the right-hand side of Eq. (18). Adding this term to Eq. (18) and applying it the result to the right-hand side of Eq. (22) yields

$$U(q,z) = D^{(N)}\left\{1 - \frac{\langle \delta W^2 \rangle^{(N)}}{(w^{(N)})^2}\left[1 + \left(\frac{4^N z c^{(N)}}{4w^{(N)}}\right)^{1/2}\right]\right\}$$
$$- D^{(N)}\left(\frac{q^2 D^{(N)}}{z}\right)\left(\frac{4^N z c^{(N)}}{4w^{(N)}}\right)^{1/2}\left[\frac{\langle \delta C^2 \rangle^{(N)}}{(c^{(N)})^2} + 2\frac{\langle \delta C \delta C_+ \rangle^{(N)}}{(c^{(N)})^2}\right] \tag{23}$$

where C_+ indicates a nearest neighbor of C.

First consider the average, $w^{(N)}$ and the mean-squared fluctuation, $\langle \delta W^2 \rangle^{(N)}$ of W after the Nth iteration of the recursion relations. It follows from Eq. (20) that

$$\left\langle \frac{1}{W} \right\rangle' = \left\langle \frac{1}{W} \right\rangle \tag{24}$$

and

$$\left\langle \delta\left(\frac{1}{W}\right)^2 \right\rangle' = \frac{1}{2}\left\langle \delta\left(\frac{1}{W}\right)^2 \right\rangle \tag{25}$$

Higher moments of δW diminish by a factor greater than 2 with each RG transformation. Thus, in the limit of large N, we obtain

$$w^{(N)} \xrightarrow[N\to\infty]{} \left[\langle W^{-1} \rangle^{(N)}\right]^{-1} = \langle W^{-1} \rangle^{-1} \equiv w^* \tag{26}$$

and

$$\frac{\langle \delta W^2 \rangle^{(N)}}{(w^{(N)})^2} \xrightarrow[N\to\infty]{} \frac{\langle (\delta W^{-1})^2 \rangle^{(N)}}{(\langle W^{-1} \rangle^{(N)})^2} = \frac{w^{*2}\langle (\delta W^{-1})^2 \rangle}{2^N} \equiv \frac{\Delta_w}{2^N} \tag{27}$$

A similar analysis can be applied to the average and second cumulants of C and C_+. It follows from Eq. (21) and the translational invariance of the average that

$$c^{(N)} = c \equiv c^* \tag{28}$$

and

$$\frac{\langle(\delta C)^2\rangle^{(N)}}{(c^{(N)})^2} + 2\frac{\langle \delta C \delta C_+ \rangle^{(N)}}{(c^{(N)})^2} \xrightarrow[N\to\infty]{} \frac{1}{2^N}\frac{\langle(\delta C)^2\rangle}{c^{*2}} \equiv \frac{1}{2^N}\Delta_c \tag{29}$$

Putting Eqs. (26)–(29) in the perturbation expansion, Eq. (23), and taking the limit $N \to \infty$ yields

$$U(q,z) = D^*\left[1 - \left(\frac{zc^*}{4w^*}\right)^{1/2}\Delta_w\right] - D^*\left(\frac{q^2D^*}{z}\right)\left(\frac{zc^*}{4w^*}\right)^{1/2}\Delta_c \tag{30}$$

where $D^* = w^*l^2/c^*$ is the physical diffusion coefficient.

Notice that the fluctuation correction to the diffusion coefficient which appeared in Eq. (23) vanishes from the final result as $N \to \infty$ since it is proportional to 2^{-N}. For the same reason, we can show that Eq. (30) is an exact asymptotic expansion in z in the following sense. Instead of the variables q and z choose variable $\xi \equiv q^2/z$ and z. Then, for fixed ξ, Eq. (30) becomes exact as $z \to 0$ for arbitrary values of Δ_w and Δ_c. To see this, consider the effect of the additional terms in the perturbation expansion which are as large as or larger than $z^{1/2}$ for small z and fixed ξ. These terms have coefficients which involve higher moments of δW or δC and for large N they diminish faster than $1/2^N$ and drop out of the series. Thus we claim to have included in Eq. (30) all terms which are as large or larger than $z^{1/2}$ for $z \to 0$ and ξ fixed.

4. DISCUSSION

We have seen that the presence of static disorder in a one-dimensional random walk leads to non-Markovian macroscopic behavior described by the generalized transport coefficient, $U(q,z)$, given in Eq. (30). Fluctuations in the transition rates (W) lead to a non-Markovian diffusion coefficient with a $z^{1/2}$ ($t^{-3/2}$) long time tail whereas fluctuations in the site energies (C) lead to a non-Markovian Burnett coefficient with a $z^{-1/2}(t^{-1/2})$ long time tail. Since the $t^{-1/2}$ tail is nonintegrable, disorder in the C's leads to a breakdown in the usual, time-independent gradient expansion of the transport law for the system.

In Section 2 we chose local equilibrium initial conditions and found that the relaxation of an ensemble is described by a response function

which differs from the average Green's function. The effect of disorder in the C's on the average Green's function was calculated in Ref. 1 and leads to a $z^{1/2}$ long time tail rather than the $z^{1/2}(q^2/z)$ term found here. The present result for the response function corresponds to the usual macroscopic experiment and is in agreement with the results of an exact calculation by Haus *et al.*[10] and a mode-coupling calculation of van Beijeren *et al.*[11]

We used a renormalization group technique to extend perturbation theory to the case of strong disorder. The RG method essentially converts the perturbation expansion in the strength of the disorder into an asymptotic small z expansion of which we have found the first two terms.

APPENDIX

In this appendix we derive the recursion relations Eqs. (20) and (21). A similar and more detailed derivation can be found in Ref. 1. Suppose that the random walk process is described by a generalized master equation with a transition matrix taking the form

$$T_{nm}(z) = \frac{W_{m(z)}}{C_m(z)}\delta_{nm+1} + \frac{W_{m-1}(z)}{C_m(z)}\delta_{nm-1} + \left[\frac{W_m(z) + W_{m-1}(z)}{C_m(z)}\right]\delta_{nm} \tag{A.1}$$

The first step is to convert to the continuous time random walk picture.[12] Let $p_n(z)$ be the Laplace transform of the probability density for making a jump from site n to site $n+1$ after waiting at site n for a time t. Similarly, let $q_n(z)$ the Laplace transform of the probability density for making a jump from site n to site $n-1$ after waiting at site n for a time t. Using the Laplace transformed waiting time distributions, the elimination of the odd-numbered sites is simply accomplished by summing the possible ways of hopping from site $2n$ to site $2n+2$ or $2n-2$ with all possible intermediate jumps to site $2n+1$ and $2n-1$. The result, after rescaling time by $1/\lambda$ and renumbering the sites, is

$$p'_n(\lambda z) = \frac{p_{2n}(z)p_{2n+1}(z)}{1 - p_{2n}(z)q_{2n+1}(z) - q_{2n}(z)p_{2n-1}(z)} \tag{A.2}$$

and

$$q'_n(\lambda z) = \frac{q_{2n}(z)q_{2n-1}(z)}{1 - p_{2n}(z)q_{2n+1}(z) - q_{2n}(z)p_{2n-1}(z)} \tag{A.3}$$

The numerators account for direct hops from $2n$ to $2n+2$ or $2n-2$ while the denominators are the result of summing the geometric series representing the possible jumps back and forth between $2n$ and $2n+1$ or $2n-1$.

The relation between $p_n(z)$, $q_n(z)$ and $C_n(z)$ with $W_n(z)$ is

$$p_n(z) = \frac{W_n(z)}{zC_n(z) + W_n(z) + W_{n-1}(z)} \tag{A.4}$$

$$q_n(z) = \frac{W_{n-1}(z)}{zC_n(z) + W_n(z) + W_{n-1}(z)} \tag{A.5}$$

Possible recursion relations consistent with Eq. (A2)–(A5) are

$$C'_n(\lambda z) = (2/\lambda)\left[C_{2n}(z) + p_{2n-1}(z)C_{2n-1}(z) + q_{2n+1}(z)C_{2n+1}(z)\right] \tag{A.6}$$

and

$$W'_n(\lambda z) = \frac{2W_{2n}(z)W_{2n+1}(z)}{zC_{2n+1}(z) + W_{2n+1}(z) + W_{2n}(z)} \tag{A.7}$$

From Eqs. (A.4)–(A.7), Eqs. (A.2) and (A.3) can be verified by tedious algebra. It is only ratios of $W_m(z)$ and $C_m(z)$ which appear in $T_{jk}(z)$ so that each member of the set of functions $\{C_n(z), W_n(z)\}$ can be multiplied by the same arbitrary function of z without physical consequence. This induces a freedom in choosing the recursion relations for $W_n(z)$ and $C_n(z)$. The recursion relations of Eqs. (A.6) and (A.7) have the property that they do not change the value of $C_n = C_n(0)$ or $W_n = W_n(0)$ in a uniform system where $C_n = c$ and $W_n = w$, if $\lambda = 4$. The recursion relations for C_n and W_n, Eqs. (20) and (21) follow from Eqs. (A.4)–(A.7) by setting $z = 0$. In Ref. 1, W_n was called τ_n and C_n called U_n.

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